Combinatrix

→ **Sum rule** → $n(A \cup B) = n(A) + n(B)$ if A & B are mutually exclusive
if there are n ways to do task T1 & m ways to do task T2 independent of T1 then, T = T1 or T2 has m+n ways to do

→ **product rule** → if there are n ways to do task T1 & m ways to do T2 independent of T1, then T1 and T2 has m·n ways to do

→ **Inclusion exclusion** → $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

→ **Case by case** → Mutually exclusive & exhaustive

→ **Complimentary rule** → Calculate what is not required & substract from total

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→ Splitting n distinct people into teams of different sizes $s_1, s_2, s_3 \dots s_k$

- all team sizes distinct
- Unlabelled teams.

such that $\boxed{\sum_{i=1}^{k} s_i = n}$ ⇒ $\boxed{\#\text{Ways} = \dfrac{n!}{\prod_{i=1}^{k} s_i!}}$

→ Identical team sizes — a_1 size appears m_1 times
a_2 〃 〃 m_2 times
⋮
a_r 〃 〃 m_r times

$$\boxed{\#\text{ways} = \frac{n!}{\left(\prod_{i=1}^{k} s_i!\right)\left(\prod_{j=1}^{r} m_j!\right)}}$$

Intuition: ⇒ Arrange n people in a linear permutation
First s_1 people belong to Team 1
Next s_2 〃 〃 〃 〃 2 & so on.
But within each team i $s_i!$ arrangements possible
if a size repeats m times any m! arrangement of these teams are treated equal since unlabelled teams

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→ #arrangements of n distinct items in a circle (clockwise & counterclockwise treated differently) $= (n-1)!$

→ n! linear permutation
→ over counting by a factor of n
so $\frac{n!}{n} = (n-1)!$

→ # of distinct **non negative** solutions to $x_1 + x_2 + x_3 \dots x_n = r$
Condition: $x_i \geq 0$, order matters, repetition allowed (replacement)

n boxes, r dots → #ways = $\boxed{\binom{n+r-1}{r} = \binom{n+r-1}{n-1}}$

→ # of distinct **positive** solutions to $x_1 + x_2 + x_3 + \dots x_r = n$
Condition: $x_i \geq 1$, order matters, repetition allowed (replacement)

n boxes, r−n dots → #ways = $\boxed{\binom{n-1}{r-1} = \binom{n-1}{n-r}}$

Modify $(x_1 - 1) + (x_2 - 1) + \dots (x_r - 1) = n - r$

$x_i \geq 0$	$\binom{n+k-1}{k-1}$
$x_i \geq 1$	$\binom{n-1}{k-1}$

→ No of odd 4 digit nos without repeating the digits. (First digit cannot be zero, last digit needs to be odd).

Solution

$9 \quad 9 \quad 8 \quad \cancel{5}$ wrong

start counting from left to right

Case by case

1) No zeroes in 4 digit number

$6 \times 7 \times 8 \times 5 = 1680$

2) Zero in 2nd digit position

$7 \quad 1 \quad 8 \quad 5 = 280$

3) Zero in 3rd digit

$7 \quad 8 \quad 1 \quad 5 = 280$

∴ Total = $\underline{2240}$

Guideline 1: Start counting the most restricted task, especially when applying product rule.

$8 \times 8 \times 7 \times 5 = \underline{2240}$

most restricted

2nd most restricted

→ 8 different people are to divided into 3 teams such that each team contain at least 2 persons. Find the total #ways to form such teams.

Method 1

Case I: $\;^8C_2 \; \frac{^6C_3}{2} = \frac{\overset{4}{\cancel{8}} \times 7}{\cancel{2}} \times \frac{\overset{3}{\cancel{6}} \times 5 \times \cancel{4} \times 2}{1 \times \cancel{2} \times \cancel{3} \times \cancel{2}} = 4 \times 7 \times 5 \times 2 = 280$

Case II: $\;^8C_4 \times \frac{^4C_2}{2} = \frac{\overset{2}{\cancel{8}} \times 7 \times \overset{2}{\cancel{6}} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \frac{\overset{2}{\cancel{4}} \times 3}{1 \times \cancel{2} \times \cancel{2}} = 7 \times 2 \times 5 \times 3 = 210$

Method 2

Case I. $\{2,2,4\} = \frac{8!}{2!\,2!\,4!\cdot 2!} = \frac{8 \times 7 \times 6 \times 5}{2 \times 2 \times 2} = 210$

Case II: $\{3,3,2\} = \frac{8!}{3!\,3!\,2!\,2!} \quad \frac{8 \times 7 \times \cancel{6} \times 5 \times \cancel{4}}{\cancel{6} \times \cancel{2 \times 2}} = 280$

Q) # Integers b/w 1 & 10,00,000 such that the sum of digits = 15

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$$

a $x_i \geq 0$
b $x_i \leq 9$

w/o restriction (b) $\rightarrow \binom{15+6-1}{6-1} = 15504$

Counting Violations of (b)

let $x_k \geq 10$ — $6C_1 = 6$ possible choices

$$\sum_{\substack{i=1 \\ i \neq k}}^{6} (x_i) + (x_k - 10) = 15 - 10 = 5$$

$$= \binom{5+6-1}{6-1} = 252$$

$\Rightarrow 15504 - 6 \cdot 252$
$= \boxed{13992}$

Q #ways to Arrange the letters of the word TERRORISM in a circle

Solutions

$8!$ ways to arrange

$\frac{8!}{3!}$ ways discounting RRR arrangements $= 6720$

Q) 4 boys & 4 girls. Let m = #ways they can be arranged such that Boys — odd position, girls — Even position

n = #ways " " " " " " no two girls sit together

$$m/n = \frac{1}{5C_4} = 0.2$$

$m = 4! \times 4!$

$n = 4! \times 5C_4 \times 4!$

choose 4 positions

arrange girls

arrange boys

Q) 4 Letter words where repetition allowed but not consecutive

26	25	25	25

$= 26 \times 25^3$

Q) How Many strings of characters (lower case) that have letters "x" in them?

Sol Inclusion - Exclusion $26^4 - 25^4$

Q) String of 5 ASCII characters with '@' sign atleast once (128 ASCII chara) | = #Total − # No @ symbol

$= 128^5 - 127^5$

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

= 36

P6 → $36^6 - 26^6$

P7 → $36^7 - 26^7$

P8 → $36^8 - 26^8$

EXERCISES 2.3.4. How many passwords can be created with the following constraints:

1) The password is three characters long and contains two lowercase letters and one digit in some order.

$$\frac{26 \times 26 \times 10 \times 3!}{2}$$

Q) #Ways of picking 2 cards from a deck (order matters), such that atleast one are.

Solution 1)
#Total = 52×51
unfavorable = 48×47
#favorable = $52 \times 51 - 48 \times 47$
= 396

Solution 2
First card Ace — 4×48
Second " " — 4×48
Both card Ace = 4×3
= 396

Q) How many permutations of the letters ABCDEFGH contains the string ABC

$\underset{1 \text{ element}}{ABC}$ $\underbrace{D\ E\ F\ G\ H}_{5 \text{ elements}}$ ⇒ 6!

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1. Three hardcover books and 5 paperbacks are placed on a shelf. How many ways can the books be arranged if all the hardcover books must be together and all the paperbacks must be together?

$= 3! \times 5! \times 2$

Q) How many ways to arrange 5 children such that two children (A,B) are never sitting together?

S1) #Total = 5!
unfavorable = $4! \times 2$
⇒ Total favorable = $5! - 4! \times 2 = 4! \times (5-2)$
$= 4! \times 3 = 72$

S2) Arrange remaining = 3!
choose 2 pos from 4 gaps = ${}^4C_2$
permute 2 pos = 2!

$= 3! \times 2! \times {}^4C_2$
$= 6 \times 2 \times 6$
$= 72$

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

S1) # Arrange 8 men — $8!$

Permute 5 gaps from 9 = $^9P_5$

$= 8! \times {}^9P_5$

Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

Substraction

a) #Total = $^{16}C_5$

#Unfavourable = $^9C_5$

#favourable = $^{16}C_5 - {}^9C_5$

Side note

$^7C_1 \times {}^{15}C_4$ wrong due to overcounting

b) #Total = $^{16}C_5$

#Unfavourable = $^9C_5 + {}^7C_5$

Favourable = $^{16}C_5 - {}^9C_5 - {}^7C_5$

There are 6 jobs with distinct difficulty levels, and 3 computers with distinct processing speeds. Each job is assigned to a computer such that:

- The fastest computer gets the toughest job and the slowest computer gets the easiest job.
- Every computer gets at least one job.

The number of ways in which this can be done is ___________.

C1 J1

C2

C3 J6

remaining 4 jobs distributed #Total = 3^4

#Unfavourable = 2^4

#favourable = $3^4 - 2^4$

- $^nC_r = {}^nC_{n-r}$
- $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$
- $n \cdot {}^{n-1}C_{k-1} = k \cdot {}^nC_k$ — Choose the president story proof
- $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$ — from binomial theorem

  story # n bit binary numbers $= \left(\sum_k \text{\# n bit — with } k \text{ '1's} \right)$
- $\sum_{k=0}^{n} 2^k \cdot \binom{n}{k} = (1+2)^n = 3^n$ — from binomial theorem

  story — from n people # ways to form 2 teams of any strength

  $\equiv \sum_k$ [choose k people from n give everyone choice of team]

- $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots {}^nC_n x^n$
- put $x=1 \longrightarrow 2^n = \sum_{k}^{n} {}^nC_k$
- put $x=-1 \longrightarrow 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 \dots {}^nC_n$

$$\Rightarrow \boxed{{}^nC_0 + {}^nC_2 + {}^nC_4 \dots\dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots}$$

Story: # Teams of size odd number members $=$ # Teams of size even number

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Distributing Objects into Boxes

1) Distinguishable Objects Distinguishable Boxes (DODB) $\dfrac{n!}{n_1!\, n_2!\, n_3! \dots n_k!}$ → or product rule with combination $n_1+n_2+\dots n_k = n$

multinomial notation $\binom{n!}{n_1, n_2, n_3, \dots n_k}$

1b) # Ways to distribute n distinct objects into k distinct boxes, w/o any restriction on # objects/box $= k^n$

2) Identical Objects Distinguishable Boxes (IODB) $\binom{n+k-1}{k}$

Star Bar problems
Objects$_k$ (Box-1)$_n$

2b) Each Box has atleast one object $\binom{k-1}{n-1} = \binom{k-1}{k-n}$

IODB Template applies on 5 Different Types of Problems.

1. Combination with Repetition — Select k items from n items, items can repeat, order does not matter
2. Integer Solutions of an Equation — $x_1 + x_2 + \dots x_n = k \mid x_i \geq 0 \Rightarrow$ # ways $= \binom{n+k-1}{k}$; $x_1 + x_2 + \dots x_n = n+k \mid x_i \geq 1 \Rightarrow$ # Ways $\binom{m-1}{m-n}$, $m = n+k$
3. Multiset Problem — Using n elements, how many multiset of size k can be created $\Rightarrow$ # Ways $= \binom{n+k-1}{k}$
4. Non-decreasing Integer Sequence Problem
5. Integer Composition Problem

3) Identical Objects Identical Boxes (IOIB) $\binom{n+k-1}{k} / n!$

4) Distinguishable Objects Identical Boxes (DOIB) ${}^{n+k-1}P_k / n!$

Q) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution $25C_{12} \cdot 2^{12} \cdot (-3)^{13}$

The coefficient of x^4 in the polynomial $(x-1)^3(x-2)^3$ is equal to__________.

A. 33 ✓ B. −3 C. 30 D. 21

$(x-1)(x-1)(x-1)(x-2)(x-2)(x-2)$

6 terms choose 4

$6C_4 \times x^4 \times$ — not feasible

$(x-1)^3 = x^3 - 3x^2 + 3x - 1$

$(x-2)^3 = x^3 - 6x^2 + 12x - 8$

$x^4(3+18+12) = 33x^4$

Q) Determine the coefficient of xyz^2 in $(x+y+z)^4$

Solution 1) $4C_2 \cdot 2C_1 = 6 \times 2 = 12$

Solution 2) $(x+b)^4$ xb^3 coefficient $= 4C_1$

$b = y+z$

b^3 yz^2 — coefficient $= 3C_1$

$4C_1 \times 3C_1 = 12$

Q. Counting Full Houses in Poker Game

Hand in Poker = A poker hand consists of a 5 randomly chosen cards out of a deck of 52 cards.

A hand is called "Full House" if it consists three cards of the same value and two cards of another value (e.g. 3 kings and 2 eights). How many different Full Houses are possible?

$13 \times 4C_3 \times 12 \times 4C_2$

7. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (a) there are no restrictions? (b) there must be six men and six women? (c) there must be an even number of women? (d) there must be more women than men? (e) there must be at least eight men?

a) $20C_{12}$

b) $10C_6 \cdot 10C_6$

c) $10C_2 \cdot 10C_{10} + 10C_4 \cdot 10C_8 + 10C_6 \cdot 10C_6 + 10C_8 \cdot 10C_4 + 10C_2$

d) $10C_7 \cdot 10C_5 + 10C_8 \cdot 10C_4 + 10C_9 \cdot 10C_3 + 10C_{10} \cdot 10C_2$

e) $10C_8 \cdot 10C_4 + 10C_9 \cdot 10C_3 + 10C_{10} \cdot 10C_2$

Q) How many ways to distribute 5 cards to each of 4 players from a standard deck

S1) $\dfrac{52P_{20}}{(5!)^4}$ — arranging; → order within each 5 set doesn't matter.

S2) $52C_5 \cdot 47C_5 \cdot 42C_5 \cdot 37C_5$

$= \left(\dfrac{52\times51\times50\times49\times48}{5!}\right)\times47 \cdots \left(\dfrac{37\times36\times35\times34\times33}{5!}\right) = \dfrac{52!}{32! \cdot (5!)^4}$

4. How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

$$^{15}C_1 \times {}^{14}C_2 \times {}^{12}C_3 \times {}^{9}C_4 = \frac{15 \times 14 \times 13}{1!\ 2!} \times \frac{12 \times 11 \times 10}{3!} \times \frac{9 \times 8 \times 7 \times 6}{4!}$$

$$= \frac{15!}{2!\ 3!\ 4!\ 5!}$$

arrange / permute all people

discount by group size

Q) #Ways to distribute 52 cards into 4 players (each getting 13 cards) such that the oldest players gets Queen of spades.

Sl) $$\frac{51!}{(13!)^3 \cdot 12!}$$

Q) How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

a) both the balls and boxes are labeled?

#without Restriction — 3^5

#Violations $= \underbrace{{}^3C_1 \cdot [2^5 - 2]}_{\text{only 2 Box filled}} + \underbrace{{}^3C_1}_{\text{only 1 Box filled}}$

#favourable $= 3^5 - (90 + 3) = 243 - 93 = 150$

S2)

B1	B2	B3
1	1	3
1	3	1
3	1	1
2	2	1
2	1	2
1	2	2

$\frac{5!}{3!}$

$\frac{5!}{2! \cdot 2!}$

#cases $= 3 \times \frac{5!}{3!} + 3 \times \frac{5!}{2! \cdot 2!}$

$= 3 \times 4 \times 5 + 30 \times 3$

$= 60 + 90 = 150$

Q) 5 players into 3 teams $= 3^5$

No Restriction on #Players

Q) 30 Identical object into 3 distinct boxes such that each box has atleast 5 objects.

Sl) $x_1 + x_2 + x_3 = 30$

$x_1 \geq 5$ $\quad u \geq 0$ $\quad u = x_1 - 5$

$x_2 \geq 5$ $\quad v \geq 0$ $\quad v = x_2 - 5$

$x_3 \geq 5$ $\quad w \geq 1$ $\quad w = x_3 - 5$

$\Rightarrow$ #Ways $= \binom{17}{2}$

Q : How many ways are there to distribute 20 identical chocolate bars and 15 identical sticks of gum to 5 children?

$$\binom{24}{4} \cdot \binom{19}{4}$$

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3. A bakery sells exactly 5 kinds of cookies including chocolate-chip cookies and shortbread cookies. Assuming that the bakery has at least 10 cookies of each kind, how many different options are there for a box of 10 cookies if we require that the box contains at least 1 chocolate-chip cookie and at least 2 shortbread cookies?

$$\binom{11}{7} = \binom{11}{4}$$

Q : Equation: x + y + z = 10

How many Integer solutions are there for the above equation

if x >= 1, y >= -2, z >=0 ?

$x' = x-1$ | $y' = y+2$

$x \geq 1$ | $y \geq -2$

$x' \geq 0$ | $y' \geq 0$

$x' + y' + z = 11$

#Ways $= \binom{13}{11} = \binom{13}{2}$

How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

#Ways $= \binom{14}{11} = \binom{14}{3}$

Q) $X_1 + X_2 + X_3 + X_4 + X_5 = 20$ | X_1, X_2, X_3, X_4, X_5 all +ve integers ≥ 0

$1 \leq X_3 \leq 3$

S1) Case by Case

$X_3 = 1 \longrightarrow \binom{18}{3}$

$X_3 = 2 \longrightarrow \binom{17}{3}$

$X_3 = 3 \longrightarrow \binom{16}{3}$

$\frac{18 \times 17 \times 16}{6} + \frac{17 \times 16 \times 15}{6} + \frac{16 \times 15 \times 14}{6}$

$= \frac{16 \times 3}{6} \left[6 \times 17 + 17 \times 5 + 5 \times 14 \right]$

$= 8 \left[17 \times 11 + 70 \right]$

$= 8 \times 257 = 2056$

S2 without $X_3 \leq 3$ restriction

#Total $= \binom{19}{4}$

#Unfavourable $X_3 > 3$

let $X_3' = X_3 - 3$

$X_3' \geq 0$

#Unfavourable $= \binom{16}{4}$

#favorable $= \binom{19}{4} - \binom{16}{4}$

$\frac{19 \times 18 \times 17 \times 16}{24} - \frac{16 \times 15 \times 14 \times 13}{24}$

$\frac{16}{24} \times 6 \left[19 \times 3 \times 17 - 5 \times 7 \times 13 \right]$

$4 \left[969 - 455 \right] = \underline{\underline{2056}}$

Q) Using elements a, b, c. How many multisets of size 5 can be created?

$\binom{7}{5}$